Uniqueness of bubble-free solution in linear rational expectations models

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On the Uniqueness of the Bubble-Free Solution in Linear Rational Expectations Models

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Abstract

One usually identifies bubble solutions to linear rational expectations models by extra components (irrelevant lags) arising in addition to market fundamentals. Although there are still many solutions relying on a minimal set of state variables, i.e., relating in equilibrium the current state of the economic system to as many lags as initial conditions, there is a conventional wisdom that the bubble-free (fundamentals) solution should be unique. This paper examines existence of endogenous stochastic sunspot fluctuations close to solutions relying on a minimal set of state variables, which provides a natural test for identifying bubble and bubble-free solutions. It turns out that only one solution is locally immune to sunspots, independently of the stability properties of the perfect foresight dynamics. In the standard saddle point configuration for this dynamics, this solution corresponds to the so-called saddle stable path.

Keywords: rational expectations, bubbles, sunspots, saddle path property.

1 Introduction

It is now well known that the rational expectations hypothesis does not pick out in general a unique equilibrium path. Consequently one usually introduces into analysis additional selection devices that give an account of the relevance of special paths. The aim of such criteria is often to rule out bubble solutions, i.e., paths that are determined in particular by traders' expectations. Although there are cases where identifying bubble solutions and bubble-free (fundamentals) solutions turns out to be not questionable, a sampling of the literature (Flood and Garber [1980], Burmeister, Flood and Garber [1983], and more recently McCallum [1999] among others) suggests that there is still not an agreement upon what should be a bubble. This is actually the purpose of the present paper to progress toward defining bubble and bubble-free solutions in linear economies where agents forecast only one period ahead, and where the number of predetermined variables is arbitrary, but fixed.

In case of explicit justifications, the most often used criterion is that of stability or non explosiveness of endogenous variables (Blanchard and Fischer [1987], Blanchard and Kahn [1980] and Sargent [1987]). From a practical point of view, attention is typically restricted to a particular configuration where this criterion provides a unique outcome, namely the so-called saddle point configuration for the perfect foresight dynamics, characterized by a number of stable roots that is equal to the number of initial conditions (the number of predetermined variables). More precisely, the model features then a continuum of paths where the endogenous variable explodes toward infinity and only one saddle stable path where it remains bounded and even converges toward the stationary state. The dynamics restricted to the saddle stable path makes the current state to depend on a number of lags that is equal to the number of predetermined variables. Because of this property, it is usually asserted that market fundamentals entirely determine the actual path of the economy and expectations do not play a role in this equilibrium. However, the stability criterion fails to select a single solution as soon as there are more stable roots than predetermined variables, i.e., in the so-called indeterminate configuration for the perfect foresight dynamics.

The minimal-state variable (hereafter MSV) criterion by McCallum [1983] is conceived to apply also in this case. It recommends to eliminate solutions where the current state relies on a number of lags larger than the number of predetermined variables, i.e., solutions that display an extra component arising in addition to the components that reflect market fundamentals. Such solutions are said to be with a bubble, given that traders' expectations necessarily matter. There is a large agreement on ruling out these solutions and focusing attention on the solutions with a minimum number of lags, i.e., the solutions such that the number of lags is equal to the number of predetermined variables. However, in general models, there still remain many solutions involving a minimal number of lags, whereas there is a conventional wisdom that there should be a unique solution termed bubble-free. Hence, an additional device is needed to identify only one solution. McCallum [1999] proposes then to introduce a subsidiary principle that is at first sight unrelated to the definition of a bubble in general models (see d'Autume [1990] for a discussion). As McCallum [1999] emphasizes, such an augmented MSV criterion "identifies a single solution that can *reasonably* [emphasis is ours] be interpreted as the unique solution that is free of bubble components, i.e., the fundamentals solution". Precisely, this MSV criterion requires that the equilibrium path involves a minimal number of lags whatever the values of the exogenous parameters are, i.e., even in degenerate cases where some of them are equal to zero. In linear models, it appears that this condition always selects a unique solution. In particular, in the saddle point configuration, the MSV solution is the equilibrium path that corresponds to the saddle stable path. In the sequel we shall call "McCallum's conjecture" the claim that the MSV solution is the (unique) solution deserving to be called bubble-free (or the fundamentals solution).

To discuss this conjecture, we study existence of self-fulfilling sunspot-like beliefs whose support stands in the immediate vicinity of the solutions with a minimal number of lags. Namely, we consider that existence of such sunspot equilibria accounts for expectations mattering close to these solutions, and therefore, in order to deserve to be bubble-free, a solution should be free of any neighboring sunspot equilibrium. Our results are then in accordance with McCallum's conjecture: sunspot fluctuations never arise close to the MSV solution, and they may occur arbitrarily close to any other solution that display a minimal number of lags. These results are shown to hold in general (univariate) linear models where agents forecast only one period ahead, and with an arbitrary number $L \geq 0$ of predetermined variables. In this framework, the dynamics with perfect foresight is locally governed by $(L+1)$ perfect foresight (growth rates) roots $\lambda_1, ..., \lambda_{L+1}$ with $|\lambda_1| < ... < |\lambda_{L+1}|$. Hence there are $(L+1)$ solutions where the current state depends on only L lags. Each one corresponds to an equilibrium path that belongs to the eigensubspace spanned by L eigenvectors associated with L among $(L+1)$ perfect foresight roots, i.e., all these paths are defined by only L coefficients. In particular, in the saddle point configuration ($|\lambda_L| < 1 < |\lambda_{L+1}|$), the saddle stable path is governed by the L roots of lowest modulus $\lambda_1, ..., \lambda_L$. In this configuration as well as in any other, McCallum [1999]'s conjecture is that this latter solution is actually the unique bubble-free solution, i.e., beliefs are not relevant in this solution while they should generically matter for paths corresponding to L other roots (and including in particular the root of largest modulus λ_{L+1}). In order to discuss this assertion, we assume that agents observe an exogenous sunspot process that does not affect fundamentals, and they hold beliefs that are correlated to the sunspot process and consist in randomizing over paths arbitrarily close to solutions with L lags, i.e., over paths defined by L coefficients arbitrarily close to the L coefficients that define solutions with a minimal number of lags. We show that (i) beliefs can never be self-fulfilling in the neighborhood of the path that is governed by the L roots of lowest modulus $\lambda_1, ..., \lambda_L$, and that (ii) for any other solution with a minimal number of lags, there always exist sunspot processes ensuring that some beliefs are self-fulfilling.

This paper is organized in the following way. In Section 2, we present our results in the simple benchmark framework also considered by McCallum [1999] where $L = 1$. Then, in Section 3, we tackle the general case where $L \geq 0$ is arbitrary. A brief summary of the results is finally given in Section 4.

2 A Preliminary Example

The reduced form we first consider supposes that the current equilibrium state is a scalar x_t linked with both the common forecast of the next state $E(x_{t+1} \mid I_t)$ (where E denotes the mean operator and I_t the information set of agents at date t) and the predetermined state x_{t-1} through the following temporary equilibrium map:

$$\gamma E(x_{t+1} \mid I_t) + x_t + \delta x_{t-1} = 0, \tag{1}$$

where the real numbers γ and δ represent the relative weights of future and past respectively. Equation (2.1) stands for a first order approximation of a temporary equilibrium dynamics in a suitable neighborhood $V(\bar{x})$ of a locally unique stationary state $\bar{x}$ whose value is normalized to zero. This formulation is general enough to encompass equilibrium conditions of simple versions of overlapping generations economies with production (Reichlin [1986]), and those of infinite horizon models with cash-in-advance constraints (Woodford [1986], Bosi and Magris [1997]). It is also commonly used as a benchmark case in the temporary equilibrium literature (Grandmont [1998], Grandmont and Laroque [1990], [1991]). It serves the purpose of McCallum [1999]. In this model, the local perfect foresight dynamics relies on two local perfect foresight roots λ_1 and λ_2 (with $|\lambda_1| < |\lambda_2|$ by definition), i.e., there are two paths along which the current state x_t is determined by only one lag x_{t-1} through a constant growth rate (factor) x_t/x_{t-1} equal to either λ_1 or λ_2. In such paths, traders' forecasts do not *a priori* matter since the number of lags that affect the current state is equal to the number of predetermined variables. The path corresponding to λ_1 (say the λ_1-path for convenience) governs the perfect foresight restricted to the saddle stable branch in the saddle point case ($|\lambda_1| < 1 < |\lambda_2|$). The issue is whether this λ_1-path is indeed the only one that is bubble-free, as claimed by McCallum [1999]. In order to tackle this problem, we build a sunspot process over growth rates arbitrarily close to the perfect foresight roots λ_1 and λ_2. The existence of the sunspot equilibria so defined provides a clear method for defining bubbles. Actually it turns out that such expectations driven fluctuations do not arise close to the λ_1-path but that they do close to the λ_2-path, independently of the stability (determinacy) properties of the local perfect dynamics. As a result, the λ_1-path is the single solution of the model (2.1) that can be termed bubble-free.

2.1 Deterministic Rational Expectations Equilibria

A local perfect foresight equilibrium is a sequence of state variables $\{x_t\}_{t=-1}^{\infty}$ associated with the initial condition x_{-1}, and such that the recursive equation (2.1) with $E(x_{t+1} \mid I_t) = x_{t+1}$ holds at all times:

$$\gamma x_{t+1} + x_t + \delta x_{t-1} = 0. \tag{2}$$

The current state may consequently be related to either one or two lags in (2.2). In the latter case, the solution is $x_t = -(1/\gamma)x_{t-1} - (\delta/\gamma)x_{t-2}$. It displays more lags than predetermined variables. It is accordingly a bubble solution. On the contrary, the state variable obeys in the former case to the law of motion $x_t = \beta x_{t-1}$ where β satisfies $\gamma\beta^2 x_{t-1} + \beta x_{t-1} + \delta x_{t-1} = 0$ for any $x_{t-1} \in V(\bar{x})$, i.e., β is a root λ_i ($i = 1, 2$) of the characteristic polynomial associated with (2.2). Throughout the paper we assume that λ_1 and λ_2 (with $|\lambda_1| < |\lambda_2|$) are real. For these two solutions $x_t = \lambda_i x_{t-1}$ $(i = 1, 2)$, the number of lags is equal to the number of predetermined variables, and the fundamentals (γ, δ) and the initial condition x_{-1} are then sufficient to determine the actual path of the economy, i.e., forecasts play *a priori* no role. None of these two paths has *a priori* special characteristic that would justify labeling it as bubble-free. Nevertheless the λ_1-path is usually presumed to be the unique solution where bubbles are absent. In particular, this claim holds true according to the MSV criterion in McCallum [1999]. Namely, in the case $\delta = 0$, i.e., if no predetermined variables enter the model, λ_1 reduces to 0 (and the λ_1-path reduces to the steady state $x_t = \bar{x}$), whereas λ_2 does not. This implies that the current state is not relied to past realizations along the λ_1-path, whereas it is along the λ_2-path. The λ_1-path is therefore the only solution displaying a minimal number of lags whatever the values γ of δ are; this is precisely the definition of the MSV solution.

2.2 Stochastic Sunspot Rational Expectations Equilibria

The purpose of this Section is to show that traders' beliefs do not matter (resp. do matter) in the immediate vicinity of the λ_1-path (resp. the λ_2-path) when $\delta \neq 0$, which provides a simple basis for the choice of bubble-free trajectories. We shall assume that agents observe a public exogenous sunspot signal with two different states $s_t = 1, 2$ at every date $t \geq 0$. The signal follows a discrete time Markov process with stationary transition probabilities. Let Π be the 2-dimensional transition matrix whose ss'th entry $\pi_{ss'}$ is the probability of sunspot signal s' at date $t + 1$ when signal is s at date t. Agents believe that rates of growth are perfectly correlated with the exogenous stochastic process. Let β_s $(s = 1, 2)$ be the guess on the rate of growth whenever signal s is observed at the outset of a given period, i.e., agents deduce from occurrence of signal s at date t that x_t should be determined according to the following law of motion:

$$x_t = \beta_s x_{t-1}. \tag{3}$$

At date t, the information set includes all past realizations of the state variable and of the sunspot signal, i.e., $I_t = \{x_{t-1}, ..., x_{-1}, s_t, ..., s_0\}$. Although I_t does not contain x_t, we will consider that agents' expectations at date t are made conditionally to x_t, i.e. agents believe that x_{t+1} will be equal to $\beta_{s'} x_t$ with probability $\pi_{ss'}$. This way of forming expectations is made for technical simplicity purposes. It influences none of our results, that bear on stationary equilibrium only (as defined below). Namely, at equilibrium, beliefs are self-fulfilling and the actual x_t is always equal to its expected value at date t, that is $\beta_s x_{t-1}$. As a result, the expected value $E(x_{t+1} \mid I_t)$ writes:

$$x^e_{t+1} = E(x_{t+1} \mid s_t = s) = \left[\sum_{s'=1}^{2} \pi_{ss'} \beta_{s'}\right] x_t \equiv \bar{\beta}_s x_t, \tag{4}$$

where $\bar{\beta}_s$ represents the (expected) average growth rate between t and $(t+1)$ conditionally to the event $s_t = s$. The actual dynamics is obtained by reintroducing expectations (2.4) into the temporary equilibrium map (2.1). If s occurs at date t, then the actual law of motion of the state variable satisfies:

$$\gamma \bar{\beta}_s x_t + x_t + \delta x_{t-1} = 0$$

$$\Leftrightarrow x_t = -\left[\delta / (1 + \gamma \bar{\beta}_s)\right] x_{t-1} \equiv \Omega_s\left(\beta_1, \beta_2\right) x_{t-1}. \tag{5}$$

We are now in a position to define a 2-state sunspot equilibrium on growth rate, hereafter denoted $SSEG(k, L)$ where k is the number of different signals of the sunspot process and L represents the number of lags taken into account by agents. In this Section, we have consequently $k = 2$ and $L = 1$.

A 2-state stationary sunspot equilibrium on growth rate (denoted a $SSEG(2,1)$) is a pair (β, Π) where β is a 2-dimensional vector (β_1, β_2) and Π is the 2-dimensional stochastic matrix that triggers beliefs of traders, such that (i) $\beta_1 \neq \beta_2$ and (ii) $\beta_s = \Omega_s(\beta_1, \beta_2)$ for $s = 1, 2$.

At a $SSEG(2,1)$, the expected growth rate β_s used in (2.3) is self-fulfilling whatever the current sunspot signal s is, i.e., β_s coincides with the actual growth rate $\Omega_s(\beta_1, \beta_2)$ given in (2.5). The economy will indurate endogenous stochastic fluctuations as soon as condition (i) is satisfied. In the case where this condition fails, one can speak of a *degenerate* $SSEG(2,1)$. Degenerate $SSEG(2,1)$ are pairs $((\lambda_s, \lambda_s), \Pi)$ where λ_s is a perfect foresight growth rate and the transition matrix Π is arbitrary: growth rate remains constant through time and beliefs are self-fulfilling, whatever the sunspot process is.

Formally speaking, we shall say that a neighborhood of a $SSEG(2,1)$, denoted $((\beta_1, \beta_2), \Pi)$, is a product set $V \times \mathcal{M}_2$, where V is a neighborhood of (β_1, β_2) for the natural product topology on $I\!R^2$ and $\mathcal{M}_2$ is the set of all the 2-dimensional stochastic matrices Π. Then, we shall say that another $SSEG(2,1)$, denoted $((\beta_1', \beta_2'), \Pi')$, is in the neighborhood of $((\beta_1, \beta_2), \Pi)$ (resp. $(\lambda_s, \lambda_{s'})$) whenever the vector (β_1', β_2') stands close to (β_1, β_2) (resp. $(\lambda_s, \lambda_{s'})$), and whatever the matrices Π and Π' are. The next result studies existence of $SSEG(2,1)$ in the neighborhood of a λ_s-path $(s = 1, 2)$, i.e., such that (β_1, β_2) stands close enough to (λ_s, λ_s).

Let $\gamma \neq 0$ and $\delta \neq 0$. Then there is a neighborhood of (λ_1, λ_1) in which there do not exist any $SSEG(2,1)$, while $SSEG(2,1)$ do exist in every neighborhood of the (λ_2, λ_2).

Let us define the map Ω from $I\!R^2$ onto $I\!R^2$ in the following way:

$$(\beta_1, \beta_2) \rightarrow \Omega(\beta_1, \beta_2) = (\Omega_1(\beta_1, \beta_2) - \beta_1, \Omega_2(\beta_1, \beta_2) - \beta_2),$$

so that a $SSEG(2,1)$ is characterized by $\Omega(\beta_1, \beta_2) = (0,0)$ and $\beta_1 \neq \beta_2$. Let $\mathbf{D\Omega}(\beta_1, \beta_2)$ be the 2-dimensional Jacobian matrix of the map Ω calculated at point (β_1, β_2). As λ_1 and λ_2 are the roots of the characteristic polynomial corresponding to (2.2), $\gamma/\delta = 1/\lambda_1\lambda_2$. This identity and some computations lead to:

$$\mathbf{D\Omega}(\lambda_s, \lambda_s) = \frac{\lambda_s^2}{\lambda_1\lambda_2}\Pi - \mathbf{I}_2 \quad fors = 1, 2,$$

with $\mathbf{I}_2$ the 2-dimensional identity matrix.

Notice that $\Omega(\lambda_s, \lambda_s) = (0, 0)$ for every Π. Recall that the two eigenvalues of Π are $(\pi_{11} + \pi_{22} - 1)$ and 1 (see for instance Chung [1967]). The eigenvalues of $\mathbf{D\Omega}(\lambda_s, \lambda_s)$ are therefore:

$$\begin{aligned}
\mu_1 &= \frac{\lambda_s^2}{\lambda_1 \lambda_2}(\pi_{11} + \pi_{22} - 1) - 1, \\
\mu_2 &= \frac{\lambda_s^2}{\lambda_1 \lambda_2} - 1.
\end{aligned}$$

The determinant of the Jacobian is $\det \mathbf{D\Omega}(\lambda_s, \lambda_s) \equiv \mu_1 \mu_2$. In the generic case $\lambda_1 \neq \lambda_2$, one has $\mu_2 \neq 0$, and therefore:

$$\det \mathbf{D\Omega}(\lambda_s, \lambda_s) = 0 \Leftrightarrow \pi_{11} + \pi_{22} - 1 = \frac{\lambda_1 \lambda_2}{\lambda_s^2}.$$

This last condition reduces to $\pi_{11} + \pi_{22} - 1 = \lambda_2/\lambda_1$ if $\lambda_s = \lambda_1$, and $\pi_{11} + \pi_{22} - 1 = \lambda_1/\lambda_2$ if $\lambda_s = \lambda_2$. Noticing $|\pi_{11} + \pi_{22} - 1| < 1$ shows that $\det \mathbf{D\Omega}(\lambda_s, \lambda_s) = 0$ obtains for some matrices Π if and only if $s = 2$.

For the case $\lambda_s = \lambda_1$, the Proposition results then from applying the Implicit Functions Theorem to each point $((\lambda_1, \lambda_1), \Pi)$. The precise argument requires the compacity of the set of stochastic matrices Π (as a matrix Π is characterized by π_{11} and π_{22}, this set can be identified for instance to $[0, 1]^2$). It is as follows: for every matrix Π_0, there are open neighborhoods U_{Π_0} of (λ_1, λ_1) and V_{Π_0} of Π_0 and a smooth function T_{Π_0} from V_{Π_0} onto U_{Π_0} such that

$$\forall (\beta_1, \beta_2) \in U_{\Pi_0}, \forall \Pi \in V_{\Pi_0}, \mathbf{\Omega}_\Pi(\beta_1, \beta_2) = 0 \Leftrightarrow (\beta_1, \beta_2) = T_{\Pi_0}(\Pi). \tag{6}$$

By compacity of the set of stochastic matrices Π, there is a *finite* set C of Π_0 such that $\cup_{\Pi_0 \in C} V_{\Pi_0}$ is the whole set of stochastic matrices. Hence, the family of functions T_{Π_0} for $\Pi_0 \in C$ uniquely defines a smooth function $(\beta_1, \beta_2) = T(\Pi)$ on the whole set of stochastic matrices onto the intersection $\cap_{\Pi_0 \in C} U_{\Pi_0}$. One has:

$$\forall (\beta_1, \beta_2) \in \cap_{\Pi_0 \in C} U_{\Pi_0}, \forall \Pi, \mathbf{\Omega}_\Pi(\beta_1, \beta_2) = 0 \Leftrightarrow (\beta_1, \beta_2) = T(\Pi).$$

Given that $\mathbf{\Omega}_{\Pi}(\lambda_1, \lambda_1) = 0$ holds for every Π, $T(\Pi)$ is simply equal to (λ_1, λ_1) for every Π, and there is no other (β_1, β_2) in $\cap_{\Pi_0 \in C} U_{\Pi_0}$ satisfying $\mathbf{\Omega}_{\Pi}(\beta_1, \beta_2) = 0$ for some Π. As C is finite, this set $\cap_{\Pi_0 \in C} U_{\Pi_0}$ is an (open) neighborhood of (λ_1, λ_1).

For the case $\lambda_s = \lambda_2$, there are some Π such that $\det \mathbf{D\Omega}(\lambda_2, \lambda_2) = 0$. It follows then from standard local bifurcation theory that there exist some matrices Π and (non degenerate) $SSEG(2,1)$ in the neighborhood of (λ_2, λ_2) (see Chiappori, Geoffard and Guesnerie [1992] for a general argument).

The λ_1-path should accordingly be considered as the single bubble-free solution of the model, independently of the properties of the local perfect foresight dynamics, in particular even in the indeterminate case for this dynamics ($|\lambda_1| < |\lambda_2| < 1$). The restrictions $\gamma \neq 0$ and $\delta \neq 0$ are needed in Proposition 2.2. Otherwise actual growth rates are independent of sunspot signals (see Equation (2.5)). But they are actually not stringent given, first, that $\gamma \neq 0$ merely ensures that expectations matter and, second, that the bubble-free solution is easily identified in the case $\delta = 0$ (this is the steady state).

An example of stochastic fluctuations of the state variable induced by the sunspot equilibrium is depicted in Figure [1] in the hypothetical case where $s_0 = s_1 = s_2 = 1$ and $s_3 = 2$.

$$Insert here Figure\,[1]$$

This figure highlights that the state variable is pulled out of $V(\bar{x})$ in the case $|\lambda_2| > 1$. The stability condition $|\lambda_2| < 1$ should consequently be met as far as we are concerned with situations where the state variable is bounded (for instance in order to ensure that it remains in $V(\bar{x})$). It allows us to restore the conventional link between existence of sunspot fluctuations and indeterminacy of the stationary state that appears in models without predetermined variables (see Chiappori, Geoffard and Guesnerie [1992], Drugeon and Wigniolle [1994] or Shigoka [1994] among many others). It implies, however, that fluctuations will vanish in the long run.

The next result is concerned with the issue of whether the bubble-free role of the λ_1-path is robust to a slight change in the traders' beliefs. Precisely, we now consider that agents randomize over the two perfect foresight roots λ_1 and λ_2, i.e., they hold beliefs (β_1, β_2) in the neighborhood of (λ_1, λ_2) (or (λ_2, λ_1)). We show that no such beliefs are self-fulfilling as soon as the sunspot state associated to the expected growth rate β_1 near λ_1 is persistent enough, i.e., π_{11} and $(1 - \pi_{22})$ are large enough. Without loss of generality, we turn attention to $SSEG(2, 1)$ in the neighborhood of (λ_1, λ_2) only. The case where the $SSEG(2, 1)$ is close to (λ_2, λ_1) would be treated in a similar way, simply by changing indexes.

There exists a neighborhood of $(\pi_{11}, \pi_{22}) = (1, 0)$ *such that there is a neighborhood of* (λ_1, λ_2) *including no* $SSEG(2, 1)$ *associated to a sunspot process with transition probabilities in the above neighborhood of* $(\pi_{11}, \pi_{22}) = (1, 0)$.

Using the two identities $\lambda_1\lambda_2 = \delta/\gamma$ and $\lambda_1 + \lambda_2 = -1/\gamma$, it is readily verified that the Jacobian matrix $\mathbf{D\Omega}(\lambda_1, \lambda_2)$ of the map Ω calculated at point (λ_1, λ_2) is equal to:

$$\mathbf{D\Omega}(\lambda_1, \lambda_2) = \begin{pmatrix} \omega(\lambda_1, \lambda_2, \pi_{11})\pi_{11} - 1 & \omega(\lambda_1, \lambda_2, \pi_{11})(1 - \pi_{11}) \\ \omega(\lambda_2, \lambda_1, \pi_{22})(1 - \pi_{22}) & \omega(\lambda_2, \lambda_1, \pi_{22})\pi_{22} - 1 \end{pmatrix},$$

where $\omega(\lambda_1, \lambda_2, \pi_{ss})$ is:

$$\omega(\lambda_1, \lambda_2, \pi_{ss}) = \lambda_1\lambda_2 / \left[(1 - \pi_{ss})\lambda_1 + \pi_{ss}\lambda_2\right]^2.$$

The map ω is well defined when π_{ss} is in the neighborhood of 0 or 1. For $\pi_{11} = 1$ and $\pi_{22} = 0$, one has $\mathbf{\Omega}(\lambda_1, \lambda_2) = 0$ and some computations show that $\det \mathbf{D\Omega}(\lambda_1, \lambda_2) = 1 - \lambda_1/\lambda_2$. Then, in the generic case $\lambda_1 \neq \lambda_2$, $\det \mathbf{D\Omega}(\lambda_1, \lambda_2) \neq 0$, and the Implicit Functions Theorem applied at (λ_1, λ_2) with $\pi_{11} = 1$ and $\pi_{22} = 0$ shows that there exist neighborhoods U of (λ_1, λ_2) and V of $(\pi_{11}, \pi_{22}) = (1, 0)$ such that, for every matrix Π with transition probabilities in V, the only zero of $\mathbf{\Omega}_\Pi$ in U is (λ_1, λ_2). In other words, there do not exist a $SSEG(2, 1)$ in the neighborhood of (λ_1, λ_2) associated to a matrix Π with transition probabilities in V.

Figure [2] gives an example of stochastic fluctuations of the state variable that are induced by the sunspot equilibrium on growth rate described in Proposition 2.3. Here we set $s_0 = 1$ (so that $x_0 = \beta_1 x_{-1}$), $s_1 = 2$ and $s_2 = 1$.

$$Insert here Figure\,[2]$$

A sequence of state variables sustained by some $SSEG(2,1)$ described in Proposition 2.3, remains in $V(\bar{x})$ as soon as $|\lambda_2| < 1$, i.e., in the indeterminate configuration for the perfect foresight dynamics, and it will be pulled out of $V(\bar{x})$ with probability 1 if $|\lambda_1| > 1$, i.e., in the so-called source determinate configuration for this dynamics. The purpose of the next result is to provide a condition that ensures stability in the saddle point case. Given the stochastic framework under consideration, the stability concept is a *statistic* criterion ensuring that, in the long run, x_t remains in the neighborhood of the steady state $\bar{x}$ with an arbitrary high probability.

Consider a $SSEG(2,1)$*, denoted* (β, Π)*, that sustains a sequence of stochastic realizations* $\{x_t\}_{t=-1}^{+\infty}$*. It is called "stable" if and only if, for every* $\varepsilon > 0$*, there exists a date* T *such that* $P(\forall t \geq T, |x_t - \bar{x}| \leq \varepsilon) \geq 1 - \varepsilon$*. Let* q_s *be the long run probability of the signals* s $(s = 1, 2)$ *associated with the Markov transition matrix* Π*. Then a* $SSEG(2,1)$ *is stable if and only if* $|\beta_1^{q_1} \beta_2^{q_2}| < 1$*. If this stability condition holds true, then endogenous stochastic fluctuations of the state variable are vanishing asymptotically, i.e.,* $P(\lim x_t = \bar{x}) = 1$*.*

For the case $|\beta_1^{q_1} \beta_2^{q_2}| \neq 1$, the result comes from Theorem I.15.2 in Chung [1967]. Let us consider a 2-state ergodic Markov process with state space $\{\ln|\beta_1|, \ln|\beta_2|\}$ and with transition matrix Π. Applying the theorem with $f =$*Identity* gives:

$$P\left[\lim \frac{1}{t} \sum_{\tau=0}^{t} \ln|\beta_\tau| = q_1 \ln \beta_1 + q_2 \ln \beta_2\right] = 1.$$

As $\ln|x_t/x_{-1}| = \sum_{\tau=0}^{t} \ln|\beta_\tau|$, one obtains:

$$P\left[\lim \frac{1}{t} \ln|x_t/x_{-1}| = \ln|\beta_1^{q_1} \beta_2^{q_2}|\right] = 1.$$

If $|\beta_1^{q_1}\beta_2^{q_2}| < 1$ then $P\left[\lim \ln |x_t/x_{-1}| = -\infty\right] = 1$. Hence, $P\left[\lim |x_t| = 0\right] = 1$. Otherwise, $|\beta_1^{q_1}\beta_2^{q_2}| > 1$ and $P\left[\lim \ln |x_t/x_{-1}| = +\infty\right] = 1$. Now, $P\left[\lim |x_t| = +\infty\right] = 1$. For the case $\beta_1^{q_1}\beta_2^{q_2} = 1$, the result follows from the Central Limit Theorem for Markov chains (Theorem I.16.1 in Chung [1967]). Let us consider the stochastic variable Y_n defined by: $Y_n = \sum_{t_n \leq t < t_{n+1}} \ln |\beta_s|$ where t_n is the date of the nth return to the state $\ln |\beta_1|$. The condition $q_1 \ln \beta_1 + q_2 \ln \beta_2 = 0$ implies $E(Y_n) = 0$. As $E(Y_n^2)$ differs from zero, Theorem I.14.7 in Chung [1967] is applied to get the asymptotic property:

$$t \rightarrow +\infty \lim P\left(\frac{1}{t}\sum_{\tau=0}^{t} \ln |\beta_\tau| > \sqrt{Bt}\right) > 0,$$

where the constant $B = q_1 E(Y_n^2)$ is independent of n according to I.15 in Chung [1967]. Considering again $\ln |x_t/x_{-1}| = \sum_{\tau=0}^{t} \ln |\beta_\tau|$ proves the result.

3 A general Framework

We now deal with general economies where the current state depends on the (common) forecast of the next state and also on $L \geq 1$ predetermined variables through the following map:

$$\gamma E(x_{t+1} \mid I_t) + x_t +_{l=1}^{L} \delta_l x_{t-l} = 0, \tag{7}$$

where parameter δ_l $(1 \leq l \leq L)$ represents the relative contribution to x_t of the predetermined state of period $t - l$. The dynamics with perfect foresight involves now $(L + 1)$ perfect foresight roots $\lambda_1, .., \lambda_{L+1}$ (with $|\lambda_1| < ... < |\lambda_{L+1}|$). We shall concentrate attention on equilibrium paths along which the number of lags that influence the current state is equal to the number L of predetermined variables, i.e., paths defined by L coefficients only. As a consequence such paths have *a priori* no special characteristics that would justify the label bubble-free. Then, the issue is whether the path corresponding to the L perfect foresight roots of lowest modulus $\lambda_1, .., \lambda_L$ (that is the one that corresponds to the saddle stable path in the saddle point case $|\lambda_L| < 1 < |\lambda_{L+1}|$) still deserves to be considered as the unique bubble-free solution. According to McCallum [1999]'s MSV criterion, this is the case because it is the only solution that always displays a minimal number of lags, even in the degenerate case $\delta_1 = ... = \delta_L = 0$ (this path then reduces to the steady state $x_t = \bar{x}$). In order to answer this question as we did it in the preceding Section, we build sunspot equilibria over L-dimensional vectors whose components stand arbitrarily close to the L coefficients that define each path with L lags. It turns out that the solution corresponding to the L perfect foresight roots of lowest modulus, is the unique solution that has no sunspot equilibrium in its neighborhood, independently of the properties of the local dynamics with perfect foresight.

3.1 Deterministic Rational Expectations Equilibria

The state variable perfect foresight dynamics in $V(\bar{x})$ is related to the $(L+1)$ perfect foresight roots $\lambda_1, .., \lambda_{L+1}$ of the characteristic polynomial:

$$P_x(x) = \gamma x^{L+1} + x^L +_{l=1}^{L} \delta_l x^{L-l},$$

corresponding to (3.1) under the perfect foresight hypothesis $E(x_{t+1} \mid I_t) = x_{t+1}$, namely:

$$\gamma x_{t+1} + x_t +_{l=1}^{L} \delta_l x_{t-l} = 0. \tag{8}$$

We assume again that the roots of P_x are real, with $|\lambda_1| < ... < |\lambda_{L+1}|$. A local perfect foresight equilibrium is a sequence of state variables $\{x_t\}_{t=-L}^{\infty}$ associated with initial condition $(x_{-1}, \ldots, x_{-L}) \in V(\bar{x}) \times ... \times V(\bar{x})$ and such that (3.2) holds at each period. Solutions where the current state depends on $(L+1)$ lags in equilibrium, namely:

$$x_t = -(1/\gamma)x_{t-1} -_{l=1}^{L} (\delta_l/\gamma)x_{t-1-l},$$

are bubble solutions since beliefs matter at date $t = 0$. In the sequel we focus on solutions with only L lags. They are such that when traders hold for sure that the law of motion:

$$x_t =_{l=1}^{L} \beta_l x_{t-l}, \tag{9}$$

governs the state variable behavior for every x_{t-l} $(l = 1, ..., L)$ in $V(\bar{x})$ and every t, and when traders form consequently their forecasts, i.e.:

$$E(x_{t+1} \mid I_t) =_{l=1}^{L} \beta_l x_{t+1-l}, \tag{10}$$

then the actual dynamics makes their initial guess self-fulfilling. This actual dynamics obtains once (3.4) is reintroduced into (3.1):

$$x_t = -_{l=1}^{L} \left[(\delta_l + \gamma \beta_{l+1}) / (1 + \gamma \beta_1) \right] x_{t-l}, \tag{11}$$

with the convention that $\beta_{L+1} = 0$. Then, beliefs (3.3) are self-fulfilling whenever (3.3) and (3.5) coincide, i.e.:

$$\beta_l = -(\delta_l + \gamma \beta_{l+1}) / (1 + \gamma \beta_1), \tag{12}$$

for $l = 1, ..., L$. Solutions of (3.6) will be called stationary *extended growth rates* (henceforth stationary $EGR(L)$), and denoted $\hat{\beta}^b = (\hat{\beta}_1^b, ..., \hat{\beta}_L^b)$ with the convention that $\hat{\beta}^b$ governs the perfect foresight dynamics restricted to the L-dimensional eigenspace corresponding to all the perfect foresight roots but λ_b $(b = 1, ..., L+1)$. The expression of stationary $EGR(L)$ is given in Gauthier [1999]. For the sake of completeness, it is restated in the next Lemma.

Assume that the characteristic polynomial P_x corresponding to the $(L+1)$th order difference equation (3.2) *admits $(L+1)$ real and distinct roots λ_b, $1 \leq b \leq L+1$. Let the $(L+1)$-dimensional eigenvector u_b, $1 \leq b \leq L+1$, be associated with λ_b. Finally let W_b, $1 \leq b \leq L+1$, be the L-dimensional eigensubspace spanned by all the eigenvectors except u_b. The perfect foresight dynamics of the state variable restricted to W_b writes:*

$$x_t =_{l=1}^{L} \hat{\beta}_l^b x_{t-l},$$

where the lth entry $\hat{\beta}_l^b$ of the stationary $EGR(L)$ $\hat{\beta}^b$ is:

$$\hat{\beta}_l^b = (-1)^{l+1}_{1 \leq j_1 < \cdots < j_l \leq L+1} (\lambda_{j_1} \ldots \lambda_{j_l}) \quad for all j_z \neq b, z = 1, ..., l.$$

We first transform the dynamics (3.2) into a vector first order difference equation:

$$\mathbf{x}_{t+1} = \mathbf{T}\mathbf{x}_t,$$

where $\mathbf{T}$ is the companion matrix associated with P_x and $\mathbf{x}_t \equiv (x_t, ..., x_{t-L})^T$ (the symbol T represents the transpose of the vector). One easily checks that the $(L+1)$ eigenvalues of the $(L+1)$-dimensional matrix $\mathbf{T}$ are the perfect foresight roots λ_b, $1 \leq b \leq (L+1)$, and that each λ_b is associated to the $(L+1)$-dimensional eigenvector $\mathbf{u}_b$:

$$\mathbf{u}_b \equiv \left(\lambda_b^L, \lambda_b^{L-1}, \ldots, 1\right)^T.$$

For every b, the perfect foresight trajectory that is restricted to W_b is such that $\mathbf{x}_t$ is a linear combination of all the $\mathbf{u}_{b'}$ but $\mathbf{u}_b$, i.e., $\det(\mathbf{x}_t, \mathbf{P}_{-b}) = 0$ where $\mathbf{P}_{-b}$ is the $(L+1) \times L$ matrix whose columns are all the $\mathbf{u}_{b'}$ but $\mathbf{u}_b$. Developing the determinant, this latter identity rewrites:

$$x_t =_{l=1}^{L} a_l x_{t-l},$$

where each coefficient a_l is $(-1)^{l+1} \Delta_l / \Delta_0$ and the Δ_l are minors of the $(L+1)$-dimensional matrix $(\mathbf{x}_t, \mathbf{P}_{-b})$. Notice (see Arnaudiès and Fraysse [1987]) that Δ_0 is the determinant of Vandermonde and $\Delta_l = \sigma_l(\lambda_{-b}) \Delta_0$ where $\sigma_l(\lambda_{-b})$ is the lth elementary symmetric polynomial evaluated at λ_{-b} (the L-dimensional vector whose components are all the perfect foresight roots but λ_b):

$$\sigma_l(\lambda_{-b}) = 1 \leq j_1 < \cdots < j_l \leq L+1 j_l \neq b \lambda_{j_1} \lambda_{j_2} \cdots \lambda_{j_l}. \quad (13)$$

The result follows.

There are $(L+1)$ stationary $EGR(L)$, associated to $(L+1)$ different L-dimensional eigensubspaces of the $(L+1)$-dimensional local perfect foresight dynamics (3.2). We now study whether the $\hat{\beta}^{L+1}$-path is still the unique bubble-free solution by constructing sunspot equilibria over L-dimensional vectors that stand arbitrarily close to each stationary $EGR(L)$ of the economy. This $\hat{\beta}^{L+1}$-path is associated with the L-dimensional eigenspace corresponding to all the perfect foresight roots but λ_{L+1}, and it governs the saddle stable path in the so-called saddle point configuration for the perfect foresight dynamics ($|\lambda_L| < 1 < |\lambda_{L+1}|$).

3.2 Stochastic Sunspot Rational Expectations Equilibria

Consider that agents observe a k-state discrete time Markov process associated with a k-dimensional stochastic matrix Π. When signal is s at the outset of period t, i.e., $s_t = s$ ($s = 1, ..., k$), agents believe that the current state is linked to the L previous states according to the following law of motion:

$$x_t = \sum_{l=1}^{L} \beta_l^s x_{t-l}. \tag{14}$$

In other words, they believe that the current extended growth rate $\beta(t) = (\beta_1(t), ..., \beta_L(t))$ is equal to some L-dimensional vector $\beta^s = (\beta_1^s, ..., \beta_L^s)$, and they deduce from the occurrence of signal s that the next extended growth rate $\beta(t+1)$ will be equal to $\beta^{s'}$ ($s' = 1, ..., k$) with probability $\pi_{ss'}$, where $\pi_{ss'}$ is the ss'th entry of Π. Therefore their price expectation writes:

$$E(x_{t+1} \mid I_t) = \sum_{s'=1}^{k} \pi_{ss'} \sum_{l=1}^{L} \beta_l^{s'} x_{t+1-l} = \sum_{l=1}^{L} \sum_{s'=1}^{k} \pi_{ss'} \beta_l^{s'} x_{t+1-l} \equiv \sum_{l=1}^{L} \bar{\beta}_l^s x_{t+1-l},$$

where $\bar{\beta}_l^s$ represents the average weight of x_{t+1-l} in the forecast rule when $s_t = s$. The information set I_t must accordingly be formed by the current sunspot signal $s_t = s$ and the L previous realizations x_{t-l} ($l = 1, ..., L$). The actual dynamics in state $s_t = s$ is obtained by reintroducing forecasts into the temporary equilibrium map. One gets, with the convention that $\beta_{L+1}^s = 0$:

$$\gamma \sum_{l=1}^{L} \bar{\beta}_l^s x_{t+1-l} + x_t + \sum_{l=1}^{L} \delta_l x_{t-l} = 0$$

$$\Leftrightarrow x_t = -\sum_{l=1}^{L} \left[(\gamma \bar{\beta}_{l+1}^s + \delta_l)/(\gamma \bar{\beta}_1^s + 1) \right] x_{t-l} \equiv \sum_{l=1}^{L} \Omega_l(\bar{\beta}_1^s, \bar{\beta}_{l+1}^s) x_{t-l}. \tag{15}$$

A $SSEG(k,L)$ is a kL-dimensional vector $\beta = \left(\beta^1, ...\beta^k\right)$ where β^s is a L-dimensional vector $(\beta_1^s, ..., \beta_L^s)$, and a k-dimensional stochastic matrix Π such that (i) there are s and s' such that $\beta^s \neq \beta^{s'}$, and (ii) $\beta_l^s = \Omega_l(\bar{\beta}_1^s, \bar{\beta}_{l+1}^s)$ for $l = 1, ...L$ and $s = 1, ..., k$, with the convention that $\beta_{L+1}^s = 0$ for every s.

A $SSEG(k,L)$ is accordingly a k-state sunspot equilibrium over $EGR(L)$. This is namely a situation where every initial guess β_l^s in (3.8) coincides with the actual realization $\Omega_l(\bar{\beta}_1^s, \bar{\beta}_{l+1}^s)$ in (3.9), whatever the current sunspot signal s is, i.e. beliefs about $EGR(L)$ are self-fulfilling. The $(L+1)$ stationary $EGR(L)$ may be called degenerate $SSEG(k,L)$ as, for any Π, only condition (i) fails to hold true in the above definition.

We first consider local stochastic fluctuations in the immediate vicinity of every given stationary $EGR(L)$. As in the 2 sunspot state case, we shall say that a neighborhood of a $SSEG(k,L)$ denoted (β, Π) is a product set $V \times \mathcal{M}_k$, where V is a neighborhood of the vector β in $I\!R^{kL}$ and $\mathcal{M}_k$ is the set of all the k-dimensional stochastic matrices Π. Hence, a $SSEG(k,L)$ denoted (β, Π) is in the neighborhood of a $EGR(L)$ $\hat{\beta}^b$ whenever β stands close enough to the kL-dimensional vector $(\hat{\beta}^b, ..., \hat{\beta}^b)$. The next result extends Proposition 2.2.

Consider the reduced form (3.1). Assume that $\gamma \neq 0$, i.e., expectations matter, and $\delta_L \neq 0$. Then there do exist $SSEG(k,L)$ in every neighborhood of the stationary $EGR(L)$ $\hat{\beta}^b$ for any $b \neq L+1$. On the contrary, there is a neighborhood of the stationary $EGR(L)$ $\hat{\beta}^{L+1}$ (governing perfect foresight dynamics restricted to the eigensubspace corresponding to the L perfect foresight roots of lowest modulus) in which there do not exist any $SSEG(k,L)$.

Let $\bar{\beta}$ denote the kL-dimensional vector $(\bar{\beta}_1^1, \bar{\beta}_2^1, ..., \bar{\beta}_L^1, \bar{\beta}_1^2, ..., \bar{\beta}_L^2, \bar{\beta}_1^k, ..., \bar{\beta}_L^k)$. Then linearizing the equilibrium condition $\Omega_l(\bar{\beta}_1^s, \bar{\beta}_{l+1}^s) = \beta_l^s$ ($l = 1, ...L$ and $s = 1, ..., k$) in the neighborhood of the kL-dimensional vector $(\hat{\beta}^b, ..., \hat{\beta}^b)$ leads to (notice that $\bar{\beta}$

reduces to $(\hat{\beta}^b, ..., \hat{\beta}^b)$ at the point $(\hat{\beta}^b, ..., \hat{\beta}^b)$):

$$\beta = \mathbf{F}\overline{\beta}, \tag{16}$$

where $\mathbf{F}$ is a kL-dimensional matrix equal to:

$$\mathbf{F} = \begin{pmatrix} \mathbf{F}(\hat{\beta}^b) & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \ddots & & \vdots \\ \vdots & & \ddots & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{F}(\hat{\beta}^b) \end{pmatrix} where \mathbf{F}(\hat{\beta}^b) = -\frac{\gamma}{\gamma\overline{\beta}_1^b + 1} \begin{pmatrix} \hat{\beta}_1^b & 1 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ \vdots & 0 & \cdots & 0 & 1 \\ \hat{\beta}_L^b & 0 & \cdots & \cdots & 0 \end{pmatrix},$$

with $\mathbf{0}$ the L-dimensional zero matrix. It is shown in Gauthier [1999] that the L eigenvalues of the L-dimensional matrix $\mathbf{F}(\hat{\beta}^b)$ are λ_j/λ_b for every $j \neq b$ $(j, b = 1, ..., L+1)$. Observe now that $\overline{\beta} = (\Pi \otimes \mathbf{I}_L)\,\beta$ where the symbol $\otimes$ represents the Kronecker product, and where $\mathbf{I}_L$ is the L-dimensional identity matrix. Remark also that $\mathbf{F} = \mathbf{I}_L \otimes \mathbf{F}(\hat{\beta}^b)$. As a result, (3.10) becomes:

$$\begin{aligned} \beta &= \left(\mathbf{I_L} \otimes \mathbf{F}(\hat{\beta}^{\mathbf{b}})\right) (\mathbf{\Pi} \otimes \mathbf{I_L})\,\beta \\ &\Leftrightarrow \beta = \left(\Pi \otimes \mathbf{F}(\hat{\beta}^{\mathbf{b}})\right) \beta \\ &\Leftrightarrow \left[\mathbf{I}_{kL} - \left(\Pi \otimes \mathbf{F}(\hat{\beta}^b)\right)\right] \beta = \mathbf{0}. \end{aligned}$$

Since $(\hat{\beta}^b, ..., \hat{\beta}^b)$ is a solution of this system, the same argument as the one used in the proof of Proposition 2.2 shows that there exist $SSEG(k, L)$ in the neighborhood of $(\hat{\beta}^b, ..., \hat{\beta}^b)$ if and only if:

$$\det\left[\mathbf{I}_{kL} - \left(\Pi \otimes \mathbf{F}(\hat{\beta}^b)\right)\right] = 0. \tag{17}$$

Let μ_s $(s = 1, ..., k)$ be an eigenvalue of Π. Then the eigenvalues of $\mathbf{I}_{kL} - (\Pi \otimes \mathbf{F}(\hat{\beta}^b))$ are of the form $1 - \mu_s\lambda_j/\lambda_b$ for $s = 1, ..., k$ and $j = 1, ..., L+1$ and $j \neq b$ (see Magnus and Neudecker [1988]), so that (3.11) admits a solution Π if and only if there exists λ_j, $j \neq b$, such that λ_b/λ_j is an eigenvalue of Π. Therefore, given that $|\mu_s| \leq 1$ and $|\lambda_{L+1}|$ is the root of largest modulus, (3.11) is satisfied for some Π if and only if $b \neq L+1$.

Hence our approach fits McCallum's conjecture in the general framework considered in this Section in the sense that the equilibrium path defined by $\hat{\beta}^{L+1}$ is the only one that is free of any sunspot equilibrium in its neighborhood. This result builds upon Gauthier [1999] who provides related arguments for the selection of the solution corresponding to the L roots of lowest modulus. Gauthier [1999] actually shows that this bubble-free path is the only one that is locally determinate in a perfect foresight dynamics on extended growth rates. Although Proposition 3.3 is independent of the stability (determinacy) properties of the local perfect foresight dynamics, attention should be focused only on the indeterminate configuration ($|\lambda_{L+1}| < 1$) for this dynamics, as long as one prevents the state variable from leaving $V(\bar{x})$.

$$Insert here Figure\,[3]$$

Example. Figure [3] gives an example of such sunspot equilibria. It actually represents subspaces that trigger the law of motion of the state variable in $V(\bar{x})$ in the case $L = 2$, i.e., the perfect foresight dynamics is governed by three perfect foresight roots λ_1, λ_2 and λ_3 (with $|\lambda_1| < |\lambda_2| < |\lambda_3|$). The 2-dimensional subspace W_2 is spanned by eigenvectors associated with λ_1 and λ_3. As shown in Lemma 3.1, the dynamics restricted to W_2 is: $x_t = (\lambda_1 + \lambda_3)x_{t-1} - \lambda_1\lambda_3 x_{t-2}$. It follows from Proposition 3.3 that is possible to build $SSEG(k, 2)$ close to W_2. Here $k = 2$ so that these equilibria are defined by the same 2-dimensional stochastic matrix Π and two different 2-dimensional vectors (β_1^s, β_2^s) for $s = 1, 2$. Both vectors stand arbitrarily close to $(\lambda_1 + \lambda_3, -\lambda_1\lambda_3)$. They define the 2-dimensional subspaces E_1 and E_2 respectively. The state variable will alternate between E_1 and E_2 according to the current sunspot signal. In Figure [4], we depict the change in the value of the state variable for $s_0 = 1$ and $s_1 = 2$.

$$Insert here Figure\,[4]$$

As in our preliminary example we now ask whether the bubble-free role of the path defined by $\hat{\beta}^{L+1}$ will be maintained in the case where agents randomize over different stationary $EGR(L)$. For simplicity, we assume that $k = L+1$, i.e., all the stationary $EGR(L)$ enter the support of the beliefs. Precisely, we consider that traders hold beliefs β in the neighborhood of $\left(\hat{\beta}^1, ..., \hat{\beta}^{L+1}\right)$. The next result extends Proposition 2.3: we show that there is no $SSEG(L+1, L)$ as soon as the sunspot state associated to the expected growth rate β^{L+1} near $\hat{\beta}^{L+1}$ is persistent enough, i.e., every $\pi_{s(L+1)}$ is large enough.

There exists a neighborhood of $\left(\pi_{1(L+1)}, ..., \pi_{(L+1)(L+1)}\right) = (1, ..., 1)$ *such that there is a neighborhood of* $(\lambda_1, ..., \lambda_{L+1})$ *including no* $SSEG(L+1, L)$ *associated to a sunspot process with transition probabilities in the above neighborhood of* $\left(\pi_{1(L+1)}, ..., \pi_{(L+1)(L+1)}\right) = (1, ..., 1)$.

The proof mimics the proof of Proposition 3.3. Consider the following $L(L+1)$-dimensional matrix:

$$\mathbf{G} = \begin{pmatrix} \mathbf{F}(\hat{\beta}^1) & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \ddots & & \vdots \\ \vdots & & \ddots & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{F}(\hat{\beta}^{L+1}) \end{pmatrix},$$

where the $\mathbf{F}(\hat{\beta}^b)$ are the L-dimensional matrices defined in the proof of Proposition 3.3 (notice $\bar{\beta}_1^b$ is now different from $\hat{\beta}_1^b$). There exist $SSEG(L+1, L)$ in the neighborhood of $\left(\hat{\beta}^1, ..., \hat{\beta}^{L+1}\right)$ if and only if, for some matrix Π:

$$\det\left[\mathbf{I}_{L(L+1)} - \mathbf{G}\left(\Pi \otimes \mathbf{I}_L\right)\right] = 0.$$

Notice now that:

$$\mathbf{G}\left(\Pi \otimes \mathbf{I}_L\right) = \begin{pmatrix} \pi_{11}\mathbf{F}(\hat{\beta}^1) & \cdots & \pi_{1L+1}\mathbf{F}(\hat{\beta}^1) \\ \vdots & \ddots & \vdots \\ \pi_{L+11}\mathbf{F}(\hat{\beta}^{L+1}) & \cdots & \pi_{L+1L+1}\mathbf{F}(\hat{\beta}^{L+1}) \end{pmatrix}.$$

At the point $\left(\pi_{1(L+1)}, ..., \pi_{(L+1)(L+1)}\right) = (1, ..., 1)$, this matrix reduces to:

$$\mathbf{G}\left(\Pi \otimes \mathbf{I}_L\right)_{\left(\pi_{1(L+1)},...,\pi_{(L+1)(L+1)}\right)=(1,...,1)} = \begin{pmatrix} 0 & \cdots & 0 & \mathbf{F}(\hat{\beta}^1) \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & \mathbf{F}(\hat{\beta}^{L+1}) \end{pmatrix}.$$

It is then straightforward that the eigenvalues of the transpose of this matrix (and then of the matrix itself) are 0 with multiplicity L^2 and each eigenvalue of $\mathbf{F}(\hat{\beta}^1)$ with multiplicity 1. Some computations show that the eigenvalues of $\mathbf{F}(\hat{\beta}^1)$ are $\lambda_1 / \sum_{s=1}^{L+1} \pi_{js}\lambda_s$ for $j \neq 1$ ($j = 1, ..., L+1$). Precisely, these computations are as follows: every λ_j for $j \neq 1$ satisfies the polynomial identity $\lambda_j^L = \sum_{l=1}^{L} \hat{\beta}_l^1 \lambda_j^{L-l}$ and $-\lambda_j\gamma / \left(\gamma\bar{\beta}_1^1 + 1\right)$ is therefore an eigenvalue of $\mathbf{F}(\hat{\beta}^1)$ (associated to the eigenvector $\left(\lambda_j^{L-1}, ..., \lambda_j, 1\right)$). As $\bar{\beta}_1^1 = \sum_{s=1}^{L+1} \pi_{1s}\hat{\beta}_1^s$, $\hat{\beta}_1^s = \sum_{j\neq s} \lambda_j$ and $\sum_j \lambda_j = -1/\gamma$, it follows that $-\left(\gamma\bar{\beta}_1^1 + 1\right)/\gamma = \sum_{s=1}^{L+1} \pi_{js}\lambda_s$.

Finally, as the perfect foresight roots λ_s are assumed to be distinct and larger than λ_1 in modulus, no eigenvalue of $\mathbf{F}(\hat{\beta}^1)$ is equal to 1 and no eigenvalue of $\left[\mathbf{I}_{L(L+1)} - \mathbf{G}\left(\Pi \otimes \mathbf{I}_L\right)\right]$ is equal to 0. Hence, its determinant is not equal to 0 either. Then, by continuity of the determinant with the coefficients $\pi_{ss'}$, there is a compact neighborhood of the point $\left(\pi_{1(L+1)}, ..., \pi_{(L+1)(L+1)}\right) = (1, ..., 1)$ such that the determinant $\det\left[\mathbf{I}_{L(L+1)} - \mathbf{G}\left(\Pi \otimes \mathbf{I}_L\right)\right]$ is non zero for every matrix with transition probabilities in this neighborhood. Applying the same argument as the one used in proof of Proposition 2.2 shows the result.

The purpose of the next result is to provide a condition that ensures stability in the case where the stationary state is locally determinate in the perfect foresight dynamics ($|\lambda_{L+1}| > 1$). The stability concept is the same as the one defined in Proposition 2.4.

Consider a SSEG(k, L) defined by (β, Π) that sustains a sequence of stochastic realizations $\{x_t\}_{t=-L}^{+\infty}$. Let $\mathbf{B}_s$ be the L-dimensional companion matrix associated *with the L-dimensional vector β^s:*

$$\mathbf{B}_s = \begin{pmatrix} \beta_1^s & \cdots & \cdots & \beta_L^s \\ 1 & \ddots & & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix}.$$

Let $\|\mathbf{B}_s\| = \sup_{\|\mathbf{z}\|=1} |\mathbf{B}_s\mathbf{z}|$ the norm of matrix $\mathbf{B}_s$. Let q_s be the long run probability of the signal s ($s = 1, ..., k$) corresponding to Π. Then a SSEG(k, L) is stable if $\prod_{s=1}^k \|\mathbf{B}_s\|^{q_s} < 1$. If this stability condition holds true, then endogenous stochastic fluctuations of the state variable are vanishing asymptotically, i.e. $P(\lim x_t = \bar{x}) = 1$.

When the current signal is s, the L-dimensional vector $\mathbf{x}_t = (x_t, ..., x_{t-L})$ is given by:

$$\mathbf{x}_t = \mathbf{B}_s\mathbf{x}_{t-1}.$$

Hence for an history of the sunspot process $s_0, ..., s_t$, one obtains:

$$\mathbf{x}_t = \mathbf{B}_{s_t}...\mathbf{B}_{s_0}\mathbf{x}_{-1}.$$

A standard result on matrix norms is:

$$\|\mathbf{x}_t\| \leq \|\mathbf{B}_{s_t}\| ... \|\mathbf{B}_{s_0}\| \|\mathbf{x}_{-1}\|,$$

which rewrites:

$$\ln \frac{\|\mathbf{x}_t\|}{\|\mathbf{x}_{-1}\|} \leq \sum_{\tau=0}^{t} \ln \|\mathbf{B}_{s_\tau}\|.$$

Consider then the k-state ergodic Markov process with state space $\{\ln \|\mathbf{B}_1\|, ..., \ln \|\mathbf{B}_k\|\}$ and with transition matrix Π. The Proposition follows from Theorem I.15.2 in Chung [1967] as in the 2-sunspot state case of Proposition 2.4.

4 Conclusion

The purpose of this paper was to provide a criterion allowing for the definition of bubble-free solutions in dynamic rational expectations models. We have studied whether (Markovian) sunspot-like beliefs can be self-fulfilling in the neighborhood of candidates solutions for the label bubble-free, i.e., those solutions that do not display irrelevant lags with respect to the number of initial conditions. We have shown that there is only one equilibrium path close to which the sunspot fluctuations under consideration cannot arise, and we have emphasized that the choice of this path is independent of the local properties of the perfect foresight dynamics. It is worth noticing that, as soon as the suitable dynamics with perfect foresight on (extended) growth rates is written, as done in Gauthier [1999], this existence result is in accordance with the well-known results linking existence of sunspot equilibria to determinacy properties of the (correctly chosen) perfect foresight dynamics. Finally, the unique bubble-free path belongs to the eigensubspace of the perfect foresight dynamics spanned by the L roots of lowest modulus. It is the solution identified by McCallum [1999]'s MSV criterion. It accordingly fits the conventional wisdom that the saddle stable path is the unique fundamentals solution in the saddle point configuration.

References

[1] Arnaudiès, J. M. & H. Fraysse (1987) *Cours de Mathématique, Algèbre*, vol. 1. Paris, France: Dunod.

[2] d'Autume, A. (1990) On the solution of linear difference equation with rational expectations. *Review of Economic Studies* 57, 677-688.

[3] Blanchard, O. & S. Fisher (1987) *Lectures on Macroeconomics*. Cambridge, MA: MIT Press.

[4] Blanchard, O. & C. Kahn (1980) The solution of linear difference models under rational expectations. *Econometrica* 48, 1305-1311.

[5] Bosi, S. & F. Magris (1997) Endogenous business cycles: capital labor substitution and liquidity constraints with increasing returns to scale. *CORE Discussion Paper* 9769.

[6] Burmeister, E. , R. Flood & P. Garber (1983) On the equivalence of solutions in rational expectations models. *Journal of Economic Dynamics and Control* 5, 311-321.

[7] Chiappori, P. A. , P.Y. Geoffard & R. Guesnerie (1992) Sunspot fluctuations around a steady state: the case of multidimensional, one-step forward looking economic models. *Econometrica* 60, 1097-1126.

[8] Chung, K. L. (1967) *Markov Chains with Stationary Transition Probabilities*. Berlin, Germany: Springer Verlag.

[9] Drugeon, J. P. & B. Wigniolle (1994) Continuous time sunspot equilibria and dynamics in a model of growth. *Journal of Economic Theory* 69, 24-52.

[10] Flood, R. & M. Garber (1980) Market fundamentals vs. price level bubbles: the first tests. *Journal of Political Economy* 88, 745-770.

[11] Gauthier, S. (1999) Determinacy and stability under learning of rational expectations equilibria. Forthcoming in *Journal of Economic Theory.*

[12] Grandmont, J. M. (1998) Expectations formation and stability of large socioeconomic systems. *Econometrica* 66, 741-783.

[13] Grandmont, J. M. & G. Laroque (1990) Stability, expectations and predetermined variables. In P. Champsaur et al. (eds.), *Essays in Honor of Edmond Malinvaud*, vol. 1 Cambridge, MA: MIT press.

[14] Grandmont, J. M. & G. Laroque (1991) Economic dynamics with learning: some instability examples. In W.A. Barnett et al. (eds.), *Equilibrium Theory and Applications: Proceedings of the Sixth International Symposium in Economic Theory and Econometrics.* Cambridge, UK: Cambridge University Press.

[15] McCallum, B. (1983) On non-uniqueness in rational expectations models: an attempt at perspective. *Journal of Monetary Economics* 11, 139-168.

[16] McCallum, B. (1999) Role of the minimal state variable criterion in rational expectations models. *NBER Working Paper* 7087.

[17] Magnus, J. & H. Neudecker (1988) *Matrix Differential Calculus with Applications in Statistics and Econometrics.* Wiley Series in Probability and Mathematical Statistics. New York, NY: Wiley.

[18] Reichlin, P. (1986) Equilibrium cycles in an overlapping generations economy with production. *Journal of Economic Theory* 40, 89-102.

[19] Sargent, T. (1987) *Macroeconomic Theory.* New York, NY: Academic Press.

[20] Shigoka, T. (1994) A Note on Woodford's conjecture: constructing stationary sunspot equilibria in a continuous time model. *Journal of Economic Theory* 64, 531-540.

[21] Woodford, M. (1986) Stationary sunspot equilibria in a financed constrained economy. *Journal of Economic Theory* 40, 128-137.